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Two-Dimensional Coulomb Systems on a Surface of Constant Negative Curvature

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We study the equilibrium statistical mechanics of classical two-dimensional Coulomb systems living on a pseudosphere (an infinite surface of constant negative curvature). The Coulomb potential created by one point charge exists and goes to zero at infinity. The pressure can be expanded as a series in integer powers of the density (the virial expansion). The correlation functions have a thermodynamic limit, and remarkably that limit is the same one for the Coulomb interaction and some other interaction law. However, special care is needed for defining a thermodynamic limit of the free energy density. There are sum rules expressing the property of perfect screening. These generic properties can be checked on the Debye–Hückel approximation, and on two exactly solvable models, the one-component plasma and the two-component plasma, at some special temperature.

KEY WORDS: Pseudosphere; two-dimensional Coulomb systems; Coulomb potential; virial expansion; screening; exactly solvable models.

1. INTRODUCTION

How the properties of a system are affected by the curvature of the space in which the system lives is a question which arises in general relativity. This is an incentive for studying simple models.

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The classical equilibrium statistical mechanics of Coulomb systems (charged particles interacting by Coulomb's law) has been much studied. Coulomb systems have interesting generic properties, related to the screening effect. For two-dimensional systems (such as particles in a plane with a logarithmic interaction), there are exactly solvable models which are useful for checking and illustrating the generic properties. It might be of some interest to investigate what are the curvature effects.

This problem has already been considered for Coulomb systems living on a sphere[1–3] or an hypersphere.[4, 5] One of the motivations for studying such systems is that they are convenient for numerical simulations, since they are finite without boundaries. However, that very finiteness has drawbacks from a fundamental point of view, since it is not possible to take the thermodynamic limit (infinite system) without suppressing the curvature at the same time; thus one cannot get information about the effect of curvature on phenomena such as screening or phase transitions pertaining to infinite systems.

In two dimensions, a surface of constant negative curvature may have the interesting feature of being both curved and infinite. The present paper is about Coulomb systems living on such a surface. In Section 2, salient properties of that surface are reviewed. The relevant Coulomb potential is discussed in Section 3. Section 4 is about general features of statistical mechanics on the curved surface. Screening effects are studied in Section 5. Then, more specific models are considered: the Debye–Hückel approximation in Section 6, the exactly solvable one-component plasma in Section 7, the exactly solvable two-component plasma in Section 8.

2. PSEUDOSPHERE

Let us recall a few properties of the surface of constant negative curvature called a pseudosphere.[6] Such a surface is a two-dimensional manifold, the entirety of which cannot be embedded in three-dimensional Euclidean space. The surface can be represented by a plane disk of radius $2a$ centered at the origin, the Poincaré disk, with a metric such that

$$ds^2 = \frac{dr^2 + r^2\, d\varphi^2}{[1 - (r^2/4a^2)]^2} \tag{2.1}$$

in terms of the polar coordinates (r, φ). The geodesics are circle arcs normal to the boundary circle of the disk. The geodesic distance from any point inside the disk to the boundary circle is infinite: the size of the surface represented by the Poincaré disk is infinite. The Gaussian curvature has the constant value $-1/a^2$ (our definition of the Gaussian curvature is such that it is $1/R^2$ for a sphere of radius R).

Another convenient set of coordinates is (τ, φ), with τ defined through

$$r = 2a \tanh \frac{\tau}{2} \tag{2.2}$$

The corresponding expression for the metric is

$$ds^2 = a^2(d\tau^2 + \sinh^2 \tau \, d\varphi^2) \tag{2.3}$$

The geodesic distance between two points (τ, φ) and (τ', φ') is given by

$$\cosh \frac{s}{a} = \cosh \tau \cosh \tau' - \sinh \tau \sinh \tau' \cos(\varphi - \varphi') \tag{2.4}$$

In particular, the geodesic distance to the origin is $a\tau$.

This representation of a pseudosphere by a Poincaré disk is very similar to the representation of a sphere of radius R by its stereographic projection from its North pole onto the plane tangent to its South pole.[2] Here, R^2 is replaced by $-a^2$, the spherical coordinate $\pi - \theta$ is replaced by τ, trigonometric functions are replaced by hyperbolic ones. An important difference is that the (finite) sphere is represented by the whole plane, while the (infinite) surface of constant negative curvature is represented by the inside of the Poincaré disk only.

It should be noted that, for a system of particles living on a surface of constant negative curvature, having a uniform density n means that the average number of particles in the area element

$$dS = \frac{r dr \, d\varphi}{[1 - (r^2/4a^2)]^2} \tag{2.5}$$

is ndS and all points are equivalent. However, in the Poincaré disk representation, there will be an apparent non-uniform density $n[1 - (r^2/4a^2)]^{-2}$ which increases up to infinity as r approaches its upper value $2a$.

Through an appropriate conformal transformation, the Poincaré disk can be mapped onto the upper half-plane, the Poincaré half-plane. This provides another, often used, representation. Here, it will be more convenient to use the Poincaré disk.

3. THE COULOMB POTENTIAL

The Laplacian is

$$\Delta = \left(1 - \frac{r^2}{4a^2}\right)^2 \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}\right]$$

$$= \frac{1}{a^2}\left[\frac{1}{\sinh \tau}\frac{\partial}{\partial \tau}\sinh \tau \frac{\partial}{\partial \tau} + \frac{1}{\sinh^2 \tau}\frac{\partial^2}{\partial \varphi^2}\right] \tag{3.1}$$

The Coulomb potential $v(s)$ at a geodesic distance s from a unit point charge is defined by

$$\Delta v(s) = -2\pi\, \delta^{(2)}(s) \tag{3.2}$$

together with the boundary condition that $v(s)$ vanishes at infinity. In the present paper, unless otherwise specified, $\delta^{(2)}$ is the Dirac distribution on the curved manifold, i.e. such that

$$\int \delta^{(2)}(s)\, dS = 1 \tag{3.3}$$

where dS is an area element defined with the metric (2.1). The solution of (3.2) exists and is

$$v(s) = -\log \tanh \frac{s}{2a} \tag{3.4}$$

This result should be contrasted with the usual two-dimensional Coulomb potential in a flat space, $-\log(s/\ell)$, which cannot be made to vanish at infinity, whatever the choice of the constant length scale ℓ might be.

One should also note that, on a sphere, (3.2) has no solution, and one resorts to the choice[1] $-\log \sin(s/2R)$ which is the Coulomb potential created by a $+1$ point charge *and* a -1 charge uniformly spread on the sphere, or alternatively the choice[7] $-\log \tan(s/2R)$ which is the Coulomb potential created by a $+1$ point charge *and* a -1 point charge at the antipodal point. The difference is, of course, that, on a sphere, the field lines originating from a single point charge converge into another singularity at the antipodal point, while, on a pseudosphere, the field lines originating from a point charge spread away to infinity.

4. STATISTICAL MECHANICS

We now address the problem of building the classical equilibrium statistical mechanics of an infinite system of charged particles living on a pseudosphere. Two particles of charges q_i and q_j interact through the Coulomb potential $-q_i q_j \log \tanh(s_{ij}/2a)$, where s_{ij} is the geodesic distance between these particles. We shall also consider the one-component plasma model, made of one species of positive particles of charge q and a uniformly negatively charged background. What are the generic properties to be expected?

4.1. Thermodynamic Functions

On a pseudosphere, there are problems with the definition of the thermodynamic functions. For instance, for defining the free energy density, the standard procedure would be to compute the free energy F in a domain of finite area S and to take the limit of F/S as the domain becomes infinite, keeping the particle number density n fixed. In the case of a flat system, the limit does not depend on the boundary conditions, because the boundary effects are of order $S^{1/2}$ while the bulk effects are of order S. However, in the present case of a pseudosphere, the length L of the boundary and the area S are of the same order. Indeed, a "geodesic disk" of radius s, i.e., the set of points which have a geodesic distance to some center smaller than s, has an area $S = 4\pi a^2 \sinh^2(s/2a)$ while the length of its boundary is $L = 2\pi a \sinh(s/a)$. As $s \to \infty$, both S and L are of order $\exp(s/a)$. Therefore, the boundary contributions to F are important, and there is no unique thermodynamic limit of F/S.

There are nevertheless reasonable ways of defining a free energy density or other thermodynamic functions. Some of these approaches will be described in the following. Yet, it should be kept in mind that it is essential to get rid of the boundary effects.

4.2. The Virial Expansion

It is well-known that, for flat conducting Coulomb systems, the pressure cannot be expanded in integer powers of the density; the pressure is not an analytical function of the density in a neighborhood of the origin. For those systems which, at low temperature, go into a dielectric phase through a Kosterlitz–Thouless transition, the non-analyticity can be used as a signature of the conducting phase.[8, 9]

It will now be shown that, on a pseudosphere, things appear to be different. There are strong indications that the virial expansion of the pressure in powers of the density exists for Coulomb systems which however will be shown in Sections 6 and 7 to be conductors.

At temperature T (as usual, we set $\beta = 1/k_B T$ with k_B the Boltzmann constant), the virial expansion of the pressure p with respect to the total number density n would be

$$\beta p = n + \sum_{k=2}^{\infty} B_k n^k \tag{4.1}$$

For simplicity, we shall only discuss in detail the second virial coefficient B_2, for two models.

4.2.1. Two-Component Plasma. This is a model made of two species of particles of charges $\pm q$, interacting through the Coulomb potential (3.4). The second virial coefficient is

$$B_2 = -\tfrac{1}{4} \int [e^{-\beta q^2 v(s)} + e^{\beta q^2 v(s)} - 2]\, dS \tag{4.2}$$

where we have regrouped the Mayer bonds for like and unlike particles. The area element is

$$dS = 2\pi a \sinh\left(\frac{s}{a}\right) ds \tag{4.3}$$

At large distances s, the square bracket in (4.2) behaves like $4(\beta q^2)^2 \exp(-2s/a)$, and therefore the integral converges in spite of the $\sinh(s/a)$ in dS (some short-range potential should be added, if necessary, for making the integral convergent at short distances). This is to be contrasted with what happens for a flat system where B_2 diverges.

We conjecture that a similar analysis would show the higher-order virial coefficients to be also finite.

When $\beta q^2 < 2$, no short-range potential is needed for making (4.2) convergent. With $v(s)$ the pure Coulomb potential (3.4), B_2 can be explicitly computed. Through the change of variable $t = \tanh^2(s/2a)$ and an integration by parts, one obtains

$$\begin{aligned} B_2 &= -\frac{\pi}{2} \beta q^2 a^2 \int_0^1 \frac{t^{-\beta q^2/2} - t^{\beta q^2/2}}{1-t}\, dt \\ &= -\frac{\pi}{2} \beta q^2 a^2 \left[\psi\left(1 + \frac{\beta q^2}{2}\right) - \psi\left(1 - \frac{\beta q^2}{2}\right)\right] \end{aligned} \tag{4.4}$$

where ψ is the logarithmic derivative of the gamma function, which has an integral representation[10] which provides the second equality in (4.4).

4.2.2. One-Component Plasma. This is a model made of one species of particles of charge q and a uniform background charged with the opposite sign. The second virial coefficient now is

$$B_2 = -\tfrac{1}{2} \int [e^{-\beta q^2 v(s)} - 1 + \beta q^2 v(s)]\, dS \tag{4.5}$$

where we have added to the Mayer bond the contribution $\beta q^2 v(s)$ from the particle-background and background-background interactions. At large distances, the square bracket in (4.5) behaves like $2(\beta q^2)^2 \exp(-2s/a)$ and B_2 is finite.

The explicit calculation of B_2 can be performed through the same change of variable as above $t = \tanh^2(s/2a)$, with the result

$$\begin{aligned} B_2 &= -\pi \beta q^2 a^2 \int_0^1 \frac{1 - t^{\beta q^2/2}}{1-t}\, dt \\ &= -\pi \beta q^2 a^2 \left[\gamma + \psi\left(1 + \frac{\beta q^2}{2}\right)\right] \end{aligned} \tag{4.6}$$

where $\gamma = 0.577 \cdots$ is Euler's constant.

4.2.3. A Possible Definition of the Thermodynamic Functions. If indeed the virial expansion (4.1) exists, it provides, in principle, a definition of the pressure free from boundary effects. This happens because the thermodynamic limit of the virial coefficients B_k has been taken before the summation (4.1). The other thermodynamic functions such as the free energy per particle f or the internal energy per particle u can be obtained through the relations

$$p = n^2 \frac{\partial f}{\partial n}, \qquad u = \frac{\partial(\beta f)}{\partial \beta} \tag{4.7}$$

Of course, in practice, the exact resummation of the expansion (4.1) will not be feasible in general.

The low-density expansion of the excess free energy per particle f_{exc} starts as

$$\beta f_{exc} = B_2 n + \cdots \tag{4.8}$$

The corresponding expansions of the excess internal energy per particle are

— for the two-component plasma

$$u_{exc} = -\frac{\pi}{2} q^2 a^2 \left[\psi\left(1+\frac{\beta q^2}{2}\right) - \psi\left(1-\frac{\beta q^2}{2}\right) + \frac{\beta q^2}{2}\psi'\left(1+\frac{\beta q^2}{2}\right) + \frac{\beta q^2}{2}\psi'\left(1-\frac{\beta q^2}{2}\right)\right] n + \cdots \tag{4.9}$$

— for the one-component plasma

$$u_{exc} = -\pi q^2 a^2 \left[\gamma + \psi\left(1+\frac{\beta q^2}{2}\right) + \frac{\beta q^2}{2}\psi'\left(1+\frac{\beta q^2}{2}\right)\right] n + \cdots \tag{4.10}$$

4.3. Correlations

In contrast with the thermodynamic functions, the n-body densities are expected to have well-defined thermodynamic limits. This is related to the screening properties, to be discussed in Section 5: screening makes the n-body densities inside the system, far away from the boundary, insensitive to the boundary conditions.

Screening has a remarkable consequence: In the thermodynamic limit, the correlation functions are the same ones in a system with the Coulomb interaction $-\log \tanh(s/2a)$ or when this interaction is replaced by $-\log \sinh(s/2a)$. This can be shown as follows.

From (2.2) and (2.4), it its found that, in terms of the complex coordinate $z = r \exp(i\varphi)$ in the Poincaré disk, the geodesic distance s_{ij} between two particles located at z_i and z_j is such that

$$\tanh \frac{s_{ij}}{2a} = \left| \frac{(z_i - z_j)/2a}{1 - (z_i \bar{z}_j / 4a^2)} \right| \tag{4.11}$$

and

$$\sinh \frac{s_{ij}}{2a} = \frac{|z_i - z_j|/2a}{[1-(r_i/2a)^2]^{1/2}\,[1-(r_j/2a)^2]^{1/2}} \tag{4.12}$$

For a system of particles of charges q_i, with the interaction $-q_i q_j \log \tanh(s_{ij}/2a)$, the total potential energy can be written

$$H_1 = -\frac{1}{2} \sum_{i \neq j} q_i q_j \log \left| \frac{(z_i - z_j)/2a}{1 - (z_i \bar{z}_j/4a^2)} \right| + \frac{1}{2} \sum_i q_i^2 \log[1 - (r_i/2a)^2]$$
$$-\frac{1}{2} \sum_i q_i^2 \log[1 - (r_i/2a)^2] \tag{4.13}$$

The first two terms in the right-hand side of (4.13) are the potential energy of a system of particles in a flat disk of radius $2a$ with ideal conductor walls at zero potential (the second-term is the interaction of each particle with its own image). On the other hand, for a system of particles with the interaction $-q_i q_j \log \sinh(s_{ij}/2a)$, the total potential energy can be written as

$$H_2 = -\frac{1}{2} \sum_{i \neq j} q_i q_j \log \frac{|z_i - z_j|}{2a} - \frac{1}{2} \sum_i q_i^2 \log[1 - (r_i/2a)^2] \tag{4.14}$$

(the system has been assumed to be neutral, thus $\sum_i q_i = 0$ has been used). The first term in the right-hand side of (4.14) is the potential energy of a system of particles in a flat domain with plain hard walls. In both (4.13) and (4.14), the last term is a same one-body potential associated with the curvature. Comparing (4.13) and (4.14), we see that the only difference between them is the nature of the walls. Since the apparent density in the Poincaré disk (number of particles in d^2z divided by d^2z) becomes infinite at its boundary (because of the metric (2.1)), the apparent screening length will vanish at the boundary, and the correlation functions inside the Poincaré disk will be independent of the nature of the wall.

Similar consideration hold for a one-component plasma.

In the following, for computing correlation functions, choosing the interaction $-\log \tanh(s/2a)$ or $-\log \sinh(s/2a)$ will just be a matter of convenience. In some cases, it will be explicitly checked that both interactions give the same correlation functions. The interaction $-\log \sinh(s/2a)$ has the advantage of being the analog of $-\log \sin(s/2R)$ on a sphere,[1] and that makes possible to use previously developed methods.

Of course, both interactions have the same behavior $-\log(s/2a)$ in the flat system limit $a \to \infty$.

5. SCREENING AND SUM RULES

Screening is a characteristic property of conductors. Internal screening means that, at equilibrium, a particle of the system is surrounded by a polarization cloud of opposite charge. External screening means that, at equilibrium, an external charge introduced in the system is surrounded by a polarization cloud of opposite charge. For a flat system, internal screening and external screening of an infinitesimal charge, respectively, result in the two Stillinger–Lovett sum rules[11] obeyed by the correlation functions.

In this section, assuming that screening holds, we derive the generalization of these Stillinger–Lovett sum rules to the case of a Coulomb system on a pseudosphere.

5.1. Internal Screening

Let us consider a system of particles of several species α. Each particle of species α carries a charge q_α. The number density of species α is n_α. The pair correlation functions are $h_{\alpha\beta}(s)$ where s is the geodesic distance between the two particles. Internal screening is expressed by the sum rule

$$\int \sum_\beta n_\beta \, q_\beta \, h_{\alpha\beta}(s) \, dS = -q_\alpha \tag{5.1}$$

5.2. External Screening

If an external point charge Q is introduced into the system, it induces a charge density $\rho_Q(s)$ at a distance s from this external charge. External screening means that

$$\int \rho_Q(s) \, dS = -Q \tag{5.2}$$

In the case when Q is infinitesimal, linear response theory allows to transform (5.2) into a sum rule involving the correlation functions of the system in the absence of the external charge. Indeed, if the charge Q is at z, its interaction energy with the system is $\hat{H}_{int} = Q\hat{\phi}(z)$ where $\hat{\phi}(z)$ is the microscopic electric potential created at z by the system and the average induced charge density at z' is

$$\rho_Q(z') = -\beta\langle \hat{\rho}(z') \, \hat{H}_{int} \rangle = -\beta Q \langle \hat{\rho}(z') \, \hat{\phi}(z) \rangle \tag{5.3}$$

where $\hat{\rho}(z')$ is the microscopic charge density at z'. Using (5.2), one obtains the Carnie and Chan sum rule[12]

$$\beta \int \langle \hat{\rho}(z') \, \hat{\phi}(z) \rangle \, dS' = 1 \tag{5.4}$$

Since $\langle \hat{\rho}(z') \hat{\phi}(z) \rangle$ depends only on the distance s between z and z', (5.4) can be rewritten as

$$2\pi a^2 \beta \int_0^\infty \langle \hat{\rho}(0) \hat{\phi}(\tau) \rangle \sinh \tau \, d\tau = 1 \tag{5.5}$$

and transformed by two successive integrations by parts into

$$4\pi a^2 \beta \int_0^\infty \left(\log \cosh \frac{\tau}{2} \right) \left[\frac{1}{\sinh \tau} \frac{d}{d\tau} \sinh \tau \frac{d}{d\tau} \langle \hat{\rho}(0) \hat{\phi}(\tau) \rangle \right] \sinh \tau \, d\tau = 1 \tag{5.6}$$

where the Laplacian appears. From the Poisson equation,

$$\frac{1}{a^2} \frac{1}{\sinh \tau} \frac{d}{d\tau} \sinh \tau \frac{d}{d\tau} \langle \hat{\rho}(0) \hat{\phi}(\tau) \rangle = -2\pi \langle \hat{\rho}(0) \hat{\rho}(\tau) \rangle \tag{5.7}$$

Thus, we obtain the sum rule

$$4\pi a^2 \beta \int \left(\log \cosh \frac{s}{2a} \right) \rho^{(2)}(s) \, dS = -1 \tag{5.8}$$

where $\rho^{(2)}(s) = \langle \hat{\rho}(0) \hat{\rho}(\tau) \rangle$ is the charge pair correlation function. For a flat two-dimensional system ($a \to \infty$), (5.8) reduces to the usual second Stillinger–Lovett sum rule

$$\frac{\pi \beta}{2} \int_0^\infty r^2 \rho^{(2)}(r) \, 2\pi r \, dr = -1 \tag{5.9}$$

6. DEBYE–HÜCKEL APPROXIMATION

Although the Debye–Hückel approximation can be formulated for any many-component plasma, for the sake of simpler notation we consider only the case of a one-component plasma with number density n and particle charge q (the background has a charge density $-qn$). The Debye–Hückel approximation is expected to be valid in the weak coupling limit $\beta q^2 \ll 1$.

6.1. Correlations

The average electric potential $\phi(s)$ at distance s from a given particle obeys the Poisson equation

$$\Delta \phi(s) = -2\pi q [n h(s) + \delta^{(2)}(s)] \tag{6.1}$$

while the pair correlation function $h(s)$ approximately obeys the linearized Boltzmann equation

$$h(s) = -\beta q\, \phi(s) \tag{6.2}$$

Thus, we obtain the usual Debye–Hückel equation

$$(\Delta - \kappa^2)\, \phi(s) = -2\pi q \delta^{(2)}(s) \tag{6.3}$$

where the inverse Debye length κ is defined by

$$\kappa^2 = 2\pi\beta n q^2 \tag{6.4}$$

and here the Laplacian is given by (3.1). In terms of $\tau = s/a$, (6.3) becomes

$$\left(\frac{1}{\sinh \tau}\frac{d}{d\tau}\sinh \tau \frac{d}{d\tau} - \kappa^2 a^2\right)\phi = -2\pi q \delta^{(2)}(\tau) \tag{6.5}$$

The solution which vanishes at infinity is

$$\phi = q Q_\nu(\cosh \tau) \tag{6.6}$$

where Q_ν is a Legendre function of the second kind[10] and

$$\nu = -\tfrac{1}{2} + (\tfrac{1}{4} + \kappa^2 a^2)^{1/2} \tag{6.7}$$

such that $\nu(\nu+1) = \kappa^2 a^2$. The pair correlation function is given by (6.2) and the charge pair correlation function is

$$\rho^{(2)}(s) = q^2 n^2 h(s) = -\frac{1}{\beta}\left(\frac{\kappa^2}{2\pi}\right)^2 Q_\nu(\cosh \tau) \tag{6.8}$$

At large distances s, (6.8) has the exponential decay $-\exp[-\nu+1)\, s/a]$.

It can be checked that the screening sum rules (5.1) and (5.8) are satisfied. Indeed, (5.1) results from[10]

$$\int_1^\infty Q_\nu(x)\, dx = \frac{1}{\nu(\nu+1)} \tag{6.9}$$

As to (5.8), one can use the Legendre equation (6.5) for showing that $[\nu(\nu+1)]^{-1}(x^2-1)\, dQ_\nu(x)/dx$ is a primitive of $Q_\nu(x)$, and by successive integrations by parts of the left-hand side of (5.8) one reduces it to (6.9).

Thus, like in the case of a flat system, the linearized Debye–Hückel approximation exactly obeys the sum rules. This means that the system is a conductor.

6.2. Thermodynamics

From the correlation function (6.8), one can obtain all the thermodynamic functions, which are reasonably defined without boundary effects as follows.

The excess internal energy per particle u_{exc} (i.e. the potential energy per particle) is half the energy of a particle in the potential created by the *other* ones:

$$u_{exc} = \frac{1}{2} q \lim_{s \to 0} \left[\phi(s) + q \log \tanh \frac{s}{2a} \right] \tag{6.10}$$

Using the behavior[10] of Q_ν $(\cosh \tau)$ when $z = \cosh \tau = \cosh(s/a)$ is close to 1,

$$Q_\nu(z) \sim -\frac{1}{2} \log \frac{z-1}{2} - \gamma - \psi(\nu + 1) \tag{6.11}$$

one finds

$$u_{exc} = -\frac{q^2}{2} [\gamma + \psi(\nu + 1)] \tag{6.12}$$

From (6.12), where ν is a function of β and n through (6.7) and (6.4), one can obtain the other thermodynamic quantities of interest as defined by (4.7). However, the integral leading to f cannot be expressed in terms of known functions.

6.3. Limiting Cases

It can be checked that, in the flat system limit $a \to \infty$, the above results lead to the known behaviors[13]

$$\begin{aligned} &h(x) \sim -\beta q^2 K_0(\kappa r), && u_{exc} \sim -\frac{q^2}{2}(\gamma + \log \kappa a) \\ &f_{exc} \sim -\frac{q^2}{2}\left(\gamma + \log \kappa a - \frac{1}{2}\right), && p_{exc} \sim -\frac{q^2}{4} n \end{aligned} \tag{6.13}$$

The other extreme case is when the density n becomes small for a given value of the curvature radius a. The excess thermodynamic functions are analytical in the density. The excess internal energy (6.12) has its density expansion starting as

$$u_{exc} = -\frac{\pi^3}{6}\beta q^4 a^2 n + \cdots \tag{6.14}$$

in agreement with the weak-coupling limit $\beta q^2 \ll 1$ of (4.10). The other weak-coupling limits are

$$f_{exc} = -\frac{\pi^3}{12}\beta q^4 a^2 n + \cdots \tag{6.15}$$

$$\beta p_{exc} = -\frac{\pi^3}{12}(\beta q^2)^2 a^2 n^2 + \cdots \tag{6.16}$$

7. ONE-COMPONENT PLASMA AT $\beta q^2 = 2$

This is an exactly solvable model in a variety of geometries. On a pseudosphere, the pair correlation function has been recently obtained by a mapping on a field theory.[14] Here, all correlation functions will be derived by adapting the methods which have been used for the previously studied geometries.[15]

7.1. Correlation Functions

As explained in Section 4.3, one expects to find the same correlation functions either with the interaction $-\log \tanh(s/2a)$ or with the interaction $-\log \sinh(s/2a)$. It turns out that it is simpler to use the second one, as follows.

For a one-component plasma with a $-\log \sinh(s/2a)$ interaction, the potential energy can be obtained by adapting (4.14). Since

$$\Delta \log \frac{|z - z'|}{2a} = 2\pi\, \delta^{(2)}(z, z') \tag{7.1}$$

the particle-background interaction energy arising from the first term of (4.14) is $W(r)$ such that

$$\Delta W(r) = 2\pi q^2 n \tag{7.2}$$

The solution of (7.2) is, up to an additive constant,

$$W(r) = -2\pi q^2 n a^2 \log\left(1 - \frac{r^2}{4a^2}\right) \tag{7.3}$$

(choosing this W depending only on the distance r to the center of the Poincaré disk is just a matter of convenience). The second term of (4.14) is another contribution to the one-body potential. The particle-particle interaction comes from the first term of (4.14). The background-background interaction is a constant. All together, the total potential energy is

$$H = -q^2 \sum_{i>j} \log \frac{|z_i - z_j|}{2a} - \frac{q^2}{2}(4\pi n a^2 + 1) \sum_i \log\left(1 - \frac{r_i^2}{4a^2}\right) \tag{7.4}$$

up to some additive constant (actually an infinite one in the thermodynamic limit) which is irrelevant for the calculation of the correlations.

The formalism for dealing with a potential energy of the form (7.4), in the canonical ensemble, has been previously developed.[15] When $\beta q^2 = 2$, the Boltzmann factor is

$$e^{-\beta H} = C \left| \prod_{i>j} (z_i - z_j) \prod_i \left(1 - \frac{r_i^2}{4a^2}\right)^{2\pi n a^2 + 1/2} \right|^2 \tag{7.5}$$

where C is a constant. The Vandermonde determinant identity

$$\prod_{i>j} (z_i - z_j) = \det \begin{bmatrix} 1 & 1 & 1 & \cdots \\ z_1 & z_2 & z_3 & \cdots \\ z_1^2 & z_2^2 & z_3^3 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{7.6}$$

allows to rewrite (7.5) as the squared modulus of a Slater determinant

$$e^{-\beta H} = C\, |\det\{\psi_j(z_i)\}_{i,\, j = 1, 2, 3, \ldots}|^2 \tag{7.7}$$

where

$$\psi_j(z) = \left(1 - \frac{r^2}{4a^2}\right)^{2\pi n a^2 + 1/2} z^{j-1} \tag{7.8}$$

Thus, (7.7) has the form of the squared modulus of the wave function of a system of independent fermions occupying the mutually orthogonal one-particle wave functions $\psi_j(z)$, and computing the n-body densities is a standard problem. The projector on the functional space spanned by the functions ψ_j is, in the thermodynamic limit,

$$\langle z_1 | P | z_2 \rangle = \sum_{j=1}^{\infty} \frac{\psi_j(z_1)\,\overline{\psi_j(z_2)}}{\int |\psi_j(z)|^2\, dS} \tag{7.9}$$

In terms of this projector, the n-particle truncated densities are

$$\begin{aligned} n(z) &= \langle z | P | z \rangle \\ n_T^{(2)}(z_1, z_2) &= -|\langle z_1 | P | z_2 \rangle|^2 \\ n_T^{(n)}(z_1, z_2,..., z_n) &= (-)^{n+1} \sum_{(i_2, i_2 \cdots i_n)} \langle z_{i_1} | P | z_{i_2} \rangle \cdots \langle z_{i_n} | P | z_{i_1} \rangle \end{aligned} \tag{7.10}$$

where the summation runs over all cycles $(i^1, i_2 \cdots i_n)$ built with $\{1, 2,..., n\}$.

Using (2.5) and (7.8), one computes the normalization integral

$$\int |\psi_j(z)|^2\, dS = \pi(4a^2)^j \frac{\Gamma(4\pi n a^2)\,\Gamma(j)}{\Gamma(4\pi n a^2 + j)} \tag{7.11}$$

and the series (7.9) can be summed into

$$\langle z_1 | P | z_2 \rangle = n \frac{[1-(r_1^2/4a^2)]^{2\pi n a^2 + 1/2}\,[1-(r_2^2/4a^2)]^{2\pi n a^2 + 1/2}}{[1-(z_1 \bar{z}_2/4a^2)]^{4\pi n a^2 + 1}} \tag{7.12}$$

Using (7.12) in (7.10) gives the n-body densities.

The one-body density is found to be position-independent, with the value n such that the particle charge density qn is the opposite of the background charge density, as expected.

After simple manipulations using (2.4), the two-body truncated density (7.10) is found to be, in agreement with Ref. 14,

$$n^2 h(s_{12}) = n_T^{(2)}(z_1, z_2) = -\frac{n^2}{[\cosh(s_{12}/2a)]^{8\pi n a^2 + 2}} \tag{7.13}$$

where s_{12} is the geodesic distance between the two particles. This formula can be recovered from the analogous result on a sphere[1] of radius R by replacing R by ia. For large values of the distance s_{12}, (7.13) has an exponential decay.

It can be easily checked that the screening sum rules (5.1) and (5.8) are satisfied by (7.13) (for calculating the integrals, it is convenient to use $\cosh(s/2a)$ as the variable). Thus, the system is a conductor.

In the Appendix, it is checked that using the interaction $-\log\tanh(s/2a)$ gives the same correlation functions.

7.2. Magnetic Analogy

As in the case of a flat system, there is a magnetic analogy. For a quantum particle of unit charge living on the surface and submitted to a magnetic field B normal to the surface,[16] the wave functions associated with the infinitely degenerate lowest Landau level are of the form (7.8), in an appropriate gauge, with $2\pi na^2+(1/2)=Ba^2$. For a system of independent fermions filling the lowest Landau level, the two-body truncated density is just (7.13).

7.3. Thermodynamics

The excess internal energy per particle is

$$u_{exc}=\frac{n}{2}\int h(s)\, q^2 v(s)\, dS \tag{7.14}$$

where $h(s)$ is given by (7.13), $v(s)$ by (3.4), and dS by (4.3). Using the variable $t=\tanh^2(s/2a)$, after an integration by parts one obtains

$$u_{exc}=\frac{q^2}{4}\int_0^1 [(1-t)^{4\pi na^2}-1]\frac{dt}{t} \tag{7.15}$$

This is related to an integral representation[10] of the ψ function such that

$$u_{exc}=-\frac{q^2}{4}[\gamma+\psi(4\pi na^2+1)] \tag{7.16}$$

In the $a\to\infty$ limit, one correctly recovers the flat system result

$$u_{exc}\sim -\frac{q^2}{4}(\gamma+\log 4\pi na^2) \tag{7.17}$$

The other extreme case is the small-n expansion (indeed, u_{exc} is an analytic function of n around $n=0$)

$$u_{exc}=-\frac{\pi^3}{6}q^2a^2n+\cdots$$

which agrees with the result (4.10) from the virial expansion with $\beta q^2=2$.

Since u_{exc} is known only at one temperature ($\beta q^2 = 2$), it is not possible to use (4.7) for obtaining the free energy per particle and the pressure. Only the beginning of the virial expansion is available through (4.6) and (4.8):

$$f_{exc} = -\pi a^2 q^2 n + \cdots \tag{7.18}$$

$$\beta p = n - \pi \beta q^2 a^2 n^2 + \cdots \tag{7.19}$$

8. TWO-COMPONENT PLASMA AT $\beta q^2 = 2$

This is a system of two species of particles with charges q and $-q$. When $\beta q^2 = 2$, this model is exactly solvable in a variety of geometries. In the present geometry of a pseudosphere, the correlations can be calculated, in the grand-canonical ensemble, by adapting the previously used methods.[2,17]

8.1. Correlations

The correlations can be computed by noting that (4.13) or (4.14) map the system onto a flat system in the Poincaré disk. The last term of (4.13) or (4.14) gives to the Boltzmann factor $\exp(-\beta H)$ a multiplicative contribution $[1-(r_i/2a)^2]$ for each particle. Furthermore, in the computation of a partition function, the area element dS_i can be expressed in terms of the area element $d^2 z_i$ as $dS_i = [1-(r_i/2a)^2]^{-2}\, d^2 z_i$. Thus, the original system with a constant fugacity ζ maps onto a flat system with a position-dependent fugacity $\zeta[1-(r/2a)^2]^{-1}$.

For that flat system, it results from previous work[17] that the n-body densities can be expressed in terms of Green functions $\tilde{G}_{\varepsilon_1 \varepsilon_2}(z_1, z_2)$ (ε_i is the sign $\pm$ of the particle at z_i). For instance, the one-body density in the flat disk for particles of sign ε at z is $4\pi\zeta a[1-(r/2a)^2]^{-1}\, \tilde{G}_{\varepsilon\varepsilon}(z, z)$. The corresponding density on the pseudosphere has a supplementary multiplicative factor $d^2 z/dS$ and is $n_\varepsilon = 4\pi\zeta a[1-(r/2a)^2]\, G_{\varepsilon\varepsilon}(z, z)$.

Thus, in terms of

$$G_{\varepsilon_1 \varepsilon_2}(z_1, z_2) = \left(1 - \frac{r_1^2}{4a^2}\right)^{1/2} \tilde{G}_{\varepsilon_1 \varepsilon_2}(z_1, z_2) \left(1 - \frac{r_2^2}{4a^2}\right)^{1/2} \tag{8.1}$$

the one-body densities are

$$n_\varepsilon = 4\pi\zeta a G_{\varepsilon\varepsilon}(z, z) \tag{8.2a}$$

Similarly, the two-body truncated densities are

$$n^{(2)\,T}_{\varepsilon_1\varepsilon_2}(z_1, z_2) = -\varepsilon_1\varepsilon_2(4\pi\zeta a)^2\,|G_{\varepsilon_1\varepsilon_2}(z_1, z_2)|^2 \tag{8.2b}$$

and higher-order correlations are given by sums over cycles similar to the ones in (7.10), the elements of the cycles now being $4\pi\zeta a G_{\varepsilon_i\varepsilon_j}(z_i, z_j)$.

The Green functions $\tilde{G}_{\varepsilon\varepsilon'}(z, z')$ form a 2×2 matrix $\tilde{G}$. It was shown[17] that, when the potential energy is of the form (4.14), $\tilde{G}$ obeys the equation

$$\left[\not{\partial} + \frac{4\pi\zeta a}{1-(r/2a)^2}\right]\tilde{G}(z, z') = \mathbf{I}\,\delta^{(2)}_{flat}(z, z') \tag{8.3}$$

where $\not{\partial}$ is the flat Dirac operator

$$\not{\partial} = 2\begin{pmatrix} 0\,\partial_z \\ \partial_{\bar{z}}\,0 \end{pmatrix} \tag{8.4}$$

and $\mathbf{I}$ the unit 2×2 matrix; $\delta^{(2)}_{flat}$ is the usual Dirac distribution in the plane. From (8.1) and (8.3), one finds

$$(\not{D} + 4\pi\zeta a)\,G(z, z') = \mathbf{I}\,\delta^{(2)}(z, z') \tag{8.5}$$

where

$$\not{D} = \left(1 - \frac{r^2}{4a^2}\right)^{3/2}\not{\partial}\left(1 - \frac{r^2}{4a^2}\right)^{-1/2} \tag{8.6}$$

is the Dirac operator on the pseudosphere, and now $\delta^{(2)}$ is the Dirac distribution on the pseudosphere, obeying (3.3). Thus G is the Green function of $\not{D} + 4\pi\zeta a$, on the pseudosphere, and the densities have the simple expressions (8.2) in terms of G.

The operator $(1/4\pi a)\,\not{\partial}$ appeared in the formalism as the inverse of a matrix kernel

$$M(z, z') = \begin{pmatrix} 0 & \dfrac{2a}{z - z'} \\ \dfrac{2a}{\bar{z} - \bar{z}'} & 0 \end{pmatrix} \tag{8.7}$$

associated to the potential energy (4.14):

$$\frac{1}{4\pi a}\not{\partial} M(z, z') = \mathbf{I}\delta^{(2)}_{flat}(z, z') \tag{8.8}$$

If, instead of (4.14), one uses the potential energy (4.13) which is like the one of a system with ideal conductor walls, (8.7) is replaced by[18,19]

$$M(z, z') = \begin{pmatrix} \dfrac{4a^2}{4a^2 - z\bar{z}'} & \dfrac{2a}{z - z'} \\ \dfrac{2a}{\bar{z} - \bar{z}'} & \dfrac{4a^2}{4a^2 - \bar{z}z'} \end{pmatrix} \tag{8.9}$$

However, (8.8) remains valid (because $4a^2 - z\bar{z}'$ has no zero inside the Poincaré disk), and one finds for $\tilde{G}$ the same equation (8.3). It can be shown that in both cases (A.13) and (4.14), $\tilde{G}(z, z')$ should go to zero at infinity, i.e., when z goes to the boundary circle of the Poincaré disk. Therefore, the correlations are the same ones for a two-body interaction $\mp q^2 \log \sinh(s/2a)$ or $\mp q^2 \log \tanh(s/2a)$.

One element of the matrix equation (8.3) is

$$2\, \partial_{\bar{z}} \tilde{G}_{++}(z, z') + \frac{4\pi\zeta a}{1 - (r/2a)^2} \tilde{G}_{-+}(z, z') = 0 \tag{8.10}$$

By combining (8.10) with the other equation relating $\tilde{G}_{++}$ and $\tilde{G}_{-+}$, one finds an equation for $\tilde{G}_{++}$ alone:

$$\left[-4\, \partial_z \left(1 - \frac{r^2}{4a^2}\right) \partial_{\bar{z}} + \frac{(4\pi\zeta a)^2}{1 - (r/2a)^2} \right] \tilde{G}_{++}(z, z') = 4\pi\zeta a \delta^{(2)}_{flat}(z, z') \tag{8.11}$$

Similar equations hold for $\tilde{G}_{--}$ and $\tilde{G}_{+-}$. Since n_ε is expected to be position-independent and $n^{(2)T}_{\varepsilon_1\varepsilon_2}(z_1, z_2)$ to depend only on the geodesic distance s between points z_1 and z_2, for obtaining these quantities it is enough to compute $\tilde{G}(z, z')$ in the simplest case $z' = 0$. One looks for a solution of (8.11) depending only on r and one uses the variable $u = \cosh^2(s/2a) = [1 - (r/2a)^2]^{-1}$. Then, (8.11) becomes the hypergeometric equation

$$\left[u(1-u) \frac{d^2}{du^2} - u \frac{d}{du} + (4\pi\zeta a^2)^2 \right] \tilde{G}_{++} = 0, \qquad u > 1 \tag{8.12}$$

with the boundary condition that $\tilde{G}_{++}$ behaves like $-\zeta a \log(u - 1)$ as $r \to 0$ ($u \to 1$). Furthermore, $\tilde{G}_{++}$ should vanish at infinity ($u \to \infty$). The solution of (8.12) which satisfies these conditions is[10]

$$\begin{aligned} \tilde{G}_{++}(z, 0) = \zeta a &\frac{\Gamma(4\pi\zeta a^2)\, \Gamma(4\pi\zeta a^2 + 1)}{\Gamma(8\pi\zeta a^2 + 1)} \\ &\times u^{-4\pi\zeta a^2} F(4\pi\zeta a^2, 4\pi\zeta a^2 + 1; 8\pi\zeta a^2 + 1; u^{-1}) \end{aligned} \tag{8.13}$$

Using (8.10) gives[10]

$$\tilde{G}_{-+}(z,0)=e^{i\varphi}\zeta a\,\frac{\Gamma(4\pi\zeta a^2)\,\Gamma(4\pi\zeta a^2+1)}{\Gamma(8\pi\zeta a^2+1)}(1-u^{-1})^{1/2}\,u^{-4\pi\zeta a^2}$$
$$\times F(4\pi\zeta a^2+1,4\pi\zeta a^2+1;8\pi\zeta a^2+1;u^{-1}) \qquad (8.14)$$

One also finds $G_{--}(z,0)=G_{++}(z,0)$ and $G_{+-}(z,0)=-\overline{G_{-+}(z,0)}$. By using the expansions of the hypergeometric functions with respect to $1-u^{-1}$, one can check that these Green functions have the proper flat-system limits as $a\to\infty$ for a fixed value of $m=4\pi a\zeta$.

Using these results in (8.1) and (8.2) gives the one- and two-body densities. However, $G_{ee}(z,0)$ has a logarithmic divergence as $z\to 0$, and one should use $G_{ee}(\sigma,0)$ in (8.2a), where σ is a small cutoff, rather than $G_{ee}(0,0)$. This amounts to replacing the point particles by small hard disks of diameter σ. The final results are

$$n_{\pm}=\zeta(4\pi\zeta a^2)\left[-2\gamma-\psi(4\pi\zeta a^2)-\psi(4\pi\zeta a^2+1)+2\log\frac{2a}{\sigma}\right] \qquad (8.15\text{a})$$

(the expansion[10] of (8.13) with respect to $1-u^{-1}$ has been used),

$$n^{(2)T}_{++}(s)=n^{(2)T}_{--}(s)=-\zeta^2\frac{[\Gamma(4\pi\zeta a^2+1)]^4}{[\Gamma(8\pi\zeta a^2+1)]^2}\frac{1}{[\cosh(s/2a)]^{16\pi\zeta a^2+2}}$$
$$\times\left[F\left(4\pi\zeta a^2,4\pi\zeta a^2+1;8\pi\zeta a^2+1;\frac{1}{[\cosh(s/2a)]^2}\right)\right]^2 \qquad (8.15\text{b})$$

$$n^{(2)T}_{-+}(s)=n^{(2)T}_{+-}(s)=\zeta^2\frac{[\Gamma(4\pi\zeta a^2+1)]^4}{[\Gamma(8\pi\zeta a^2+1)]^2}\frac{[\tanh(s/2a)]^2}{[\cosh(s/2a)]^{16\pi\zeta a^2+2}}$$
$$\times\left[F\left(4\pi\zeta a^2+1,4\pi\zeta a^2+1;8\pi\zeta a^2+1;\frac{1}{[\cosh(s/2a)]^2}\right)\right]^2 \qquad (8.15\text{c})$$

For the calculation of higher-order n-body densities, the Green functions $G(z,z')$ with arbitrary z' are needed. They would be obtained by a conformal transformation such that 0 maps onto z', in the same way as in the case of a sphere.[2]

At small distances s, (8.15b) and (8.15c) behave like in the flat system case,[17] i.e., as $-(\log s)^2$ and $1/s^2$, respectively. At large distances, they both have an exponential decay as $\mp\exp[-(8\pi\zeta a^2+1)\,s/a]$.

The strong-curvature limit $a \to 0$ is also a low-density limit $\zeta a^2 \to 0$. In that limit (8.15b) and (8.15c) should behave like $\mp \zeta^2 \beta q^2 v \sim \mp 4\zeta^2 \exp(-s/a)$ and they do.

8.2. Thermodynamics

Since the total density $n = 2n_+$ is given as a function of the fugacity ζ by (8.15a), the pressure can be obtained by integrating $n = \zeta d(\beta p)/d\zeta$. However, the integration cannot be performed in terms of known functions for arbitrary ζ. Explicit results can be obtained in limiting cases only.

The flat system results[17] can be recovered as $a \to \infty$ for a fixed value of $m = 4\pi a\zeta$.

In the opposite limiting case $\zeta \to 0$, for a given value of a, one obtains

$$n = 2n_\pm = 2\zeta + \left(16\pi a^2 \log \frac{2a}{\sigma}\right)\zeta^2 + \cdots \tag{8.16}$$

which gives

$$\beta p = 2\zeta + \left(8\pi a^2 \log \frac{2a}{\sigma}\right)\zeta^2 + \cdots \tag{8.17}$$

and therefore

$$\beta p = n - \left(2\pi a^2 \log \frac{2a}{\sigma}\right)n^2 + \cdots \tag{8.18}$$

This is in agreement with a direct calculation of the second virial coefficient (4.2) with a lower cutoff on s at $s = \sigma$.

9. CONCLUSION

The present paper has already been summarized in the Abstract. Here are some final remarks.

Perhaps a possible way of computing a free energy without boundary effects would be to start with a finite domain of suitable shape with periodic boundary conditions and to take its thermodynamic limit. However, we have not been able to use this seemingly difficult approach.

It is likely that the Kosterlitz–Thouless transition[20] of a two-component plasma to a dielectric phase at low temperature does not exist on a pseudosphere. Since the potential $-\log \tanh(s/2a)$ now goes to zero at infinity, the original argument about a balance between energy and entropy here says that breaking a pair of oppositely charged particles always leads to a lower free energy because the entropy wins. The two-component plasma is expected to remain a conductor even at low temperature.[3]

When $\beta q^2 = 2$, i.e., when the temperature is twice the Kosterlitz–Thouless transition temperature for a flat system, the two-component plasma is a conductor even in a flat space, and *a fortiori* should be a conductor on a surface of constant negative curvature. It would be of some interest to check that conjecture by explicitly showing that the modified Stillinger–Lovett sum rule (5.8) is satisfied by the correlation functions (8.15b) and (8.15c). This is left as an open problem, because of the technical difficulty of computing the relevant integral.

Another, more fundamental, open problem is how to compute the pressure of the one-component plasma at the temperature at which the pair correlation function (7.13) is known. For a flat system, one would use the virial relation

$$p = \beta^{-1} n - \frac{n^2}{4} q^2 \int h(r) \frac{dv}{dr} r \, dS \tag{9.1}$$

Is there a generalization of this relation on a pseudosphere?

APPENDIX: MORE ON THE CORRELATION FUNCTIONS OF THE ONE-COMPONENT PLASMA

The correlation functions of Section 7.1 can also be obtained when using the interaction $-\log \tanh(s/2a)$, by adapting the grand canonical formalism used in Section 8.1 for the two-component plasma.

Equation (8.3) could have been obtained from the integral equation[18, 19]

$$\tilde{G}(z, z') + \int M(z, z'') \frac{\zeta}{1 - (r''/2a)^2} \tilde{G}(z'', z') \, d^2 z'' = \frac{1}{4\pi a} M(z, z') \tag{A.1}$$

where M is the matrix (8.7). Indeed, applying $\not{\partial}$ to both sides of (A.1) gives (8.3) when (8.8) is taken into account.

[3] An opposite case when there is no Kosterlitz–Thouless phase transition is a two-component plasma on the surface of a cylinder of finite radius and infinite length.[21] At large separations, the interaction potential behaves like the distance, the system behaves like a one-dimensional two-component plasma and is always in a dielectric phase.[22]

Equation (A.1) can be adapted to the case of a one-component plasma by keeping only the $++$ components of the matrices $\tilde{G}$ and M (from now on, $\tilde{G}_{++}$ will be called $\tilde{G}$). Furthermore, the particle-background interaction (7.3) brings into the position-dependent fugacity a supplementary multiplicative factor

$$e^{-\beta W}=e^{C}\left[1-\left(\frac{r}{2a}\right)^{2}\right]^{4\pi na^2} \tag{A.2}$$

where C comes from the additive constant in the particle-background interaction. Here C is important, having the property $C\to+\infty$ in the thermodynamic limit. With these modifications, (A.1) becomes

$$\tilde{G}(z,z')+\zeta e^{C}\int\frac{1}{1-(z\bar{z}''/4a^2)}\left(1-\frac{r''^2}{4a^2}\right)^{4\pi na^2-1}\tilde{G}(z'',z')\,d^2z''$$
$$=\frac{1}{4\pi a[1-(z\bar{z}'/4a^2)]} \tag{A.3}$$

From (A.3), it results that $\tilde{G}(z,z')$ is an analytic function of z. For simplicity, we shall consider only the case $z'=0$. If one looks for a solution of circular symmetry, $\tilde{G}(z,0)$ is necessarily a constant, and (A.3) becomes

$$\tilde{G}\left[1+\frac{\zeta e^{C}}{n}\right]=\frac{1}{4\pi a} \tag{A.4}$$

The n-body densities involve $4\pi a\zeta e^{C}\tilde{G}$ which, in the thermodynamic limit $C\to\infty$, becomes the background density n, independent of the fugacity ζ. Including in (8.2a) and (8.2b) the multiplicative factor (A.2) for each fugacity gives n for the particle density, and

$$n_T^{(2)}(z,0)=-n^2\left[1-\left(\frac{r}{2a}\right)^{2}\right]^{4\pi na^2+1} \tag{A.5}$$

for the two-body truncated density in agreement with (7.13).

The above calculation illustrates that, in a grand-canonical approach to the one-component plasma with a fixed background, the bulk properties are fugacity-independent.

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